JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **176**, 390–403 (1993)

A Diagonal Hyperbolic System for Mappings with Prescribed Principal Strains*

JULIAN GEVIRTZ

Facultad de Matemáticas, Universidad Católica de Chile, Casilla 306, Santiago 22, Chile

Submitted by Avner Friedman

Received April 16, 1991

We derive a diagonal hyperbolic system of $n(n+1)(n+2)/2$ equations satisfied by the component functions of an n-dimensional mapping with given distinct principal strains together with certain combinations of their first and second order derivatives. This permits us to treat, in an elementary manner and under weak regularity assumptions on the initial data, the natural initial value problem for such mappings studied previously by D. DeTurck and D. Yang (*Duke Math. J.* **51**, 1984, 243–260) and throws light on the mechanism behind the formation of singularities. © 1993 Academic Press, Inc.

INTRODUCTION

Let $D \subset \mathbf{R}^n$ be a domain and let $f\colon D \to \mathbf{R}^n$ be a C^1 mapping whose Jacobian transformation J_f has nowhere vanishing determinant. The principal stretches $M_1(x) \leqslant M_2(x) \leqslant \cdots \leqslant M_n(x)$ of f at x are the positive square roots of the eigenvalues of the positive definite linear transformation $J_f^*(x)\, J_f(x)$, and as such are uniquely determined. We henceforth, in slight abuse of standard terminology, refer to these n functions as the principal strain functions of f. Gasqui [Gas] addressed the question of existence in the small of mappings with prescribed principal strain functions and showed that the answer was affirmative in the case in which these functions are real-analytic and satisfy

$$0 < M_1(x) < \cdots < M_n(x). \tag{0.1}$$

* This research was supported in part by a grant from the Dirección de Investigación of the Catholic University of Chile.

0022-247X/93 $5.00

DeTurck and Yang [DY] showed that the answer remains affirmative if, in addition to (0.1), one only assumes that these functions are C^∞ and, moreover, showed that a natural initial value problem, to be described in Section 1, has a solution in the C^∞ category; their analysis was carried out in the more general context of Riemannian manifolds. The basis of Gasqui's work was the Cartan–Kähler theorem, whereas DeTurck and Yang obtained their results by showing that the linearization of the first order partial differential equations which govern these mappings form a diagonal hyperbolic system (a general definition of this term is given at the beginning of Section 3) and then applying the Nash–Moser implicit function theorem.

In this article we present an alternate approach to the initial value problem considered in [DY] which, in addition to being of an elementary character, permits considerable relaxation of the smoothness hypotheses; it is based on the fact that if the linearization of a first order system is diagonal hyperbolic, then the initial value problem for this system is equivalent to one corresponding to a diagonal hyperbolic system in which the unknown functions are suitable combinations of the original unknown functions and their first and second order derivatives. We begin in Section 1 with an explanation of the notation and terminology we use and a precise description of DeTurck–Yang initial value problem. Although to follow these authors we work in the general context of Riemannian manifolds, we do so in a way that avoids reference to differential geometric terminology and machinery. Instead of making direct application of the above-mentioned fact to the linearization given in [DY], we give in Section 2 a self-contained derivation of a diagonal hyperbolic system in $n(n+1)(n+2)/2$ unknown functions which is satisfied by the n component functions of a mapping with given principal strains and certain geometrically significant combinations their first and second derivatives. In Section 3 we briefly outline enough of the method of characteristics as applied to diagonal hyperbolic systems to conclude that the initial value problem for this system is well-posed and, with appropriate data, is equivalent to the initial value problem considered in [DY]. As a consequence it follows that the DeTurck–Yang initial value problem has a local solution if the principal strain functions (assumed throughout to satisfy (0.1)) are sufficiently differentiable (C^4 is enough) and the initial mapping function has Lipschitz continuous first derivatives. Furthermore, the treatment given makes clear that the mechanism which impedes global existence of smooth solutions is essentially that which underlies the formation of singularities in nonlinear one-dimensional wave phenomena.

1. The DeTurck–Yang Initial Value Problem

Apart from notational differences and several other minor points the contents of this section are taken from [DY]. Let D and E be domains in $\mathbf{R}^n$ on which there are defined continuous $n \times n$ positive definite symmetric matrix functions $[G_{ij}]$ and $[H_{ij}]$, respectively; in other words, D and E represent coordinate neighborhoods on two Riemannian manifolds. We denote the inner products defined by these matrices by $\langle \cdot, \cdot \rangle$. Although the context should serve to eliminate any ambiguity, we often add an appropriate subscript for extra clarity. The standard inner product on $\mathbf{R}^n$ is denoted, as usual, by a dot. Throughout the paper, in making statements about all the components of an indexed quantity or to denote the entire array we omit the subscripts; thus, for example, $x = (x_1, ..., x_n)$, etc. Let $f: D \to E$ be a C^1 mapping with nonvanishing Jacobian determinant. We denote by V_G and V_H the vector space $\mathbf{R}^n$ with inner products corresponding to $G(x)$ and $H(f(x))$. The principal strain functions M_i of f are the positive square roots of the positive definite transformation $J_f^* J_f$ of V_G onto itself. (Here $J_f(x) : V_G \to V_H$.) We always assume that the M_i satisfy (0.1), the relevance of this assumption being explained in [DY, Remark 3.4]. It is easy to see that these M_i are the principal strain functions of f if and only if there exist fields $X_1, ..., X_n$ and $Y_1, ..., Y_n$ on D such that

$$J_f X_j = M_j Y_j, \qquad 1 \leqslant j \leqslant n \tag{1.1}$$

and

$$\langle X_i, X_j \rangle_{G(x)} = \delta_{ij} = \langle Y_i, Y_j \rangle_{H(f(x))}, \qquad 1 \leqslant i, j \leqslant n. \tag{1.2}$$

Although the Y_i are in essence tangent vectors to the manifold coordinatized by E at the point corresponding to $f(x)$, they are formally vector fields on D. Assuming that D is simply connected, these fields may be taken to be continuous, and if $(X_1, ..., X_N)$ is one such choice for this n-tuple of continuous fields, then the only other possibilities are of the form $(\pm X_1, ..., \pm X_N)$.

Let D_c denote the intersection of D with the hyperplane $x_n = c$ and assume that the principal strains $K_1, ..., K_{n-1}$ of the restriction f_c of f to D_c satisfy

$$0 < M_1 < K_1 < M_2 < \cdots < K_{n-1} < M_n \tag{1.3}$$

at all points of D_c. That these K_i are the principal strains of f_c means, of course, that there exist continuous vector fields $U_1, ..., U_n, V_1, ..., V_n$ which satisfy

$$J_{f_c} U_j = K_j V_j, \qquad 1 \leqslant j \leqslant n-1 \tag{1.4}$$

and

$$\langle U_i, U_j \rangle_{G(x)} = \delta_{ij} = \langle V_i, V_j \rangle_{H(f(x))}, \qquad 1 \leqslant i, j \leqslant n. \tag{1.5}$$

Here again the fields U_i are determined to within a factor of ± 1.

In what follows we will use the symbol AN to denote any specific array of analytic functions of some or all of the $3n^2$ quantities

$$\begin{gathered} \frac{\partial f_i}{\partial x_j},\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n-1; \\ G_{ij},\ H_{ij},\ 1 \leqslant i,\ j \leqslant n;\ M_i,\ 1 \leqslant i \leqslant n. \end{gathered} \tag{1.6}$$

For what we do, there is no need to calculate the exact form of the analytic functions which arise. In any such expression AN it is understood that H_{ij} is to be evaluated at $f(x)$.

Let B denote the $n \times n$ symmetric matrix whose main diagonal entries are $K_1^2, ..., K_{n-1}^2, b_n$ and whose last row and column have entries $b_1, ..., b_n$, all other entries being 0. As shown in [DY] (see the proof of Proposition 3.8, in particular the last line on page 253 and relation (3.12)), B has eigenvalues $M_1^2, ..., M_n^2$ if and only if $b_n = \sum_{i=1}^n M_i^2 - \sum_{i=1}^{n-1} K_i^2$ and

$$b_j = \pm\left(-\prod_{i=1}^{n} (M_i^2 - K_j^2) \Big/ \prod_{i \neq j} (K_i^2 - K_j^2)\right)^{1/2}, \qquad 1 \leqslant j \leqslant n-1.$$

We denote these 2^{n-1} possibilities for B by $B^1, B^2, ...$. Let A be the linear transformation of $\mathbf{R}^n$ defined by $AV_j = U_j$, $1 \leqslant j \leqslant n$. We note that there are two possibilities for A since after selecting $U_1, ..., U_n$ (that is, after deciding which signs we want to use) there are two possibilities for V_n compatible with (1.4) and (1.5). It is easy to see that the matrix with respect to the standard basis of each of these possibilities is of the form AN. The mapping f has principal strains $M_1, ..., M_n$ at $x \in D_c$ if and only if $[AJ_f(x)]_U^T [AJ_f(x)]_U = B^\sigma$ (where T denotes transpose), for some σ, where $[S]_U$ denotes the matrix of S with respect to the basis U. From this it follows that there are 2^n possibilities for $[J_f]_U$, each of which is of the form AN. This then implies that there are 2^n possibilities for $\partial f/\partial x_n$ each of which is again of the form AN. In summary, there are 2^n n-tuples $(F_1^\sigma, ..., F_n^\sigma)$ of analytic functions in the $3n^2$ variables listed in (1.6) such that if f has principal strains M_i in D and for each c the principal strains K_i of the restriction of f to D_c satisfy (1.3), then for some σ

$$\frac{\partial f_j}{\partial x_n} = F_j^\sigma\left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_{n-1}}, G(x), H(f(x))\right), \qquad 1 \leqslant j \leqslant n \tag{1.7}$$

holds in D. We also point out that no matter which σ is relevant to a given such f, the associated vector fields X_i all have nonvanishing component in the x_n-direction. This follows since the n-tuple of components of X_i with respect to the basis U is an eigenvector of B^σ corresponding to the eigenvalue M_i^2, and the final element of such an n-tuple cannot be 0.

Let D_0 be a neighborhood in $\mathbf{R}^{n-1} = \{x \in \mathrm{R}^n : x_n = 0\}$ and let $\bar{x} = (x_1, ..., x_{n-1})$. The DeTurck–Yang initial value problem requires finding solutions f^σ of these 2^n systems in a neighborhood of D_0 in $\mathbf{R}^n$ which satisfy $f^\sigma(\bar{x}, 0) = \phi(\bar{x})$, where the principal strains of $\phi: D_0 \to \mathbf{R}^n$ are assumed to satisfy (1.3). If the initial mapping ϕ as well as G, H, and M are analytic then because of the analyticity of the F_j^σ it follows from the Cauchy–Kowalewski theorem that each of these initial value problems has a unique analytic solution in a neighborhood of D_0.

2. Additional Equations for Mappings with Given Principal Strains

Let D, $E \subset \mathbf{R}^n$ be domains on which the matrix functions G and H are of class C^2. Let $f: D \to E$ be a C^3 mapping whose principal strain functions M_i satisfy (0.1). It is easy to see that the M_i are necessarily C^2 and that the $2n$ vector fields X_i, Y_i can be chosen to be so. We implicitly work in the standard coordinate system in $\mathbf{R}^n$ with respect to which the components of a vector field Z are denoted by Z^p, $1 \leqslant p \leqslant n$. We denote the directional derivative of u in the direction X_i by $D_i u$, that is,

$$D_i u = \nabla u \cdot X_i = \sum_p X_i^p \frac{\partial u}{\partial x_p}.$$

In this regard $D_i Z$ denotes the vector with components $D_i Z^p$. Furthermore, we define $a_{ij,k} = \langle D_k X_j, X_i \rangle_{G(x)}$ and $b_{ij,k} = \langle D_k Y_j, Y_i \rangle_{H(f(x))}$, so that

$$D_k X_j = \sum_i a_{ij,k} X_i \qquad \text{and} \qquad D_k Y_j = \sum_i b_{ij,k} Y_i. \tag{2.1}$$

For each triple $i > j > k$ we define

$$\begin{aligned} c_{ij,k} &= b_{ij,k} - (M_j/M_i)\, a_{ij,k} \\ c_{ik,j} &= b_{ik,j} - (M_i/M_k)\, a_{ik,j} \\ c_{jk,i} &= b_{jk,i} - (M_k/M_j)\, a_{jk,i}, \end{aligned}$$

and for $i > j$ we define

$$c_{ij,j} = a_{ij,j} \qquad \text{and} \qquad c_{ij,i} = a_{ij,i}.$$

The diagonal hyperbolic system we will derive will be satisfied by the n components $f_1, ..., f_n$ of f, the $2n^2$ components X_i^p, Y_i^p, $1 \leqslant i, p \leqslant n$, and the $n^2(n-1)/2$ functions $c_{ij,k}$, $1 \leqslant j < i \leqslant n$, $1 \leqslant k \leqslant n$. Since in the generality in which we are working the expressions which arise are quite complex, we merely keep track of their form instead of writing them out completely. For this purpose we define

$$\mu = \prod_i M_i, \qquad \Delta = \prod_{i>j} (M_i^2 - M_j^2),$$

and use the symbol Q to denote an expression of the form $P/(\mu\Delta)^N$, where N is a non-negative integer and P is a polynomial in the variables X_i^p, Y_i^p, M_i, $\partial M_i/\partial x_j$, $\partial G_{ij}/\partial x_k$, $\partial H_{ij}/\partial x_k$. The symbol Q_c will denote an expression of similar kind in which the variables $c_{ij,k}$ may also appear in the polynomial. In both cases, of course, we understand that $\partial H_{ij}/\partial x_k$ is to be evaluated at $f(x)$. We emphasize that all of the indicated arguments do not necessarily appear in a given expression of the form Q or Q_c, and that the exponent N may be 0. We also use the symbols Q, Q_c to denote vectors and matrices whose components are of the specified form. To avoid using an unnecessary subscript we abbreviate J_f by J.

The basis of the derivation which follows is the set of relations (1.1). We begin by applying D_k to the first half of (1.2) written more explicitly as

$$\langle X_i, X_j \rangle = \sum_{p,q} X_i^p G_{pq} X_j^q;$$

this is easily seen to yield an identity of the form

$$a_{ji,k} + a_{ij,k} = Q. \tag{2.2}$$

Since

$$\begin{aligned} D_k H_{pq}(f(x)) &= \nabla H_{pq}(f(x)) \cdot D_k f(x) = \nabla H_{pq}(f(x)) \cdot JX_k \\ &= M_k \nabla H_{pq}(f(x)) \cdot Y_k, \end{aligned}$$

we see in like manner that

$$b_{ji,k} + b_{ij,k} = Q. \tag{2.3}$$

Now, the compatibility relations for the mixed second order partial derivatives of the component functions of f can be written in the form

$$(D_k J)\, X_j = (D_j J)\, X_k, \qquad j \neq k,$$

which is easily seen to be equivalent to

$$D_k(JX_j) - J(D_k X_j) = D_j(JX_k) - J(D_j X_k), \qquad j \neq k.$$

Upon applying (1.1) we see that

$$D_k(M_j Y_j) - J(D_k X_j) = D_j(M_k Y_k) - J(D_j X_k), \qquad j \neq k,$$

and consequently that

$$\begin{aligned}(D_k M_j)\, Y_j + M_j(D_k Y_j) - J(D_k X_j)&\\ = (D_j M_k)\, Y_k + M_k(D_j Y_k) - J(D_j X_k),& \qquad j \neq k.\end{aligned}$$

If we take into account the definition (2.1) of $a_{ij,k}$ and $b_{ij,k}$, and at the same time apply (1.1), we see that this last line may be written as

$$\begin{aligned}\sum_i \{(D_k M_j)\, \delta_{ij} + M_j b_{ij,k} - M_i a_{ij,k}\}\, Y_i&\\ = \sum_i \{(D_j M_k)\, \delta_{ik} + M_k b_{ik,j} - M_i a_{ik,j}\}\, Y_i.&\end{aligned}$$

Since the Y_i are linearly independent we therefore have the following relations:

$$\begin{aligned}(D_k M_j)\, \delta_{ij} + M_j b_{ij,k} - M_i a_{ij,k}&\\ = (D_j M_k)\, \delta_{ik} + M_k b_{ik,j} - M_i a_{ik,j},& \qquad j \neq k,\ 1 \leqslant i \leqslant n. \qquad (2.4)\end{aligned}$$

First we consider the case $i = j$. It follows from (2.2) and (2.3) that $a_{jj,k}$ and $b_{jj,k}$ are both of the form Q. Since $D_k M_j = Q$, (2.4) yields $M_j a_{jk,j} - M_k b_{jk,j} = Q$, $j \neq k$. Upon interchanging the indices j and k and taking into account (2.2) and (2.3) we see that $M_k a_{jk,k} - M_j b_{jk,k} = Q$, $j \neq k$. Thus we have that

$$\begin{bmatrix} a_{ij,i} \\ a_{ij,j} \end{bmatrix} = \begin{bmatrix} M_j/M_i & 0 \\ 0 & M_i/M_j \end{bmatrix} \begin{bmatrix} b_{ij,i} \\ b_{ij,j} \end{bmatrix} + Q, \qquad i > j, \qquad (2.5)$$

and

$$\begin{bmatrix} b_{ij,i} \\ b_{ij,j} \end{bmatrix} = \begin{bmatrix} M_i/M_j & 0 \\ 0 & M_j/M_i \end{bmatrix} \begin{bmatrix} a_{ij,i} \\ a_{ij,j} \end{bmatrix} + Q, \qquad i > j, \qquad (2.6)$$

Next we consider the case in which all three indices appearing in (2.4) are different. Let $i > j > k$. Then $\delta_{ij} = \delta_{ik} = 0$, so that (2.4) reads

$$M_j b_{ij,k} - M_i a_{ij,k} = M_k b_{ik,j} - M_i a_{ik,j}. \qquad (2.7)$$

With a change of indices we also have $M_i b_{ji,k} - M_j a_{ji,k} = M_k b_{jk,i} - M_j a_{jk,i}$, and applying (2.2) and (2.3) again we get

$$M_j a_{ij,k} - M_i b_{ij,k} = M_k b_{jk,i} - M_j a_{jk,i} + Q. \qquad (2.8)$$

In similar fashion we obtain

$$M_k a_{ik,j} - M_i b_{ik,j} = M_k a_{jk,i} - M_j b_{jk,i} + Q. \tag{2.9}$$

Let $A_{ij,k}$ and $B_{ij,k}$ and $C_{ij,k}$ denote the (column) vectors whose components are $a_{ij,k}$, $a_{ik,j}$, $a_{jk,i}$ and $b_{ij,k}$, $b_{ik,j}$, $b_{jk,i}$ and $c_{ij,k}$, $c_{ik,j}$, $c_{jk,i}$, respectively. Then Eqs. (2.7), (2.8), and (2.9) can be written in matrix form as

$$\begin{bmatrix} M_j & -M_k & 0 \\ M_i & 0 & M_k \\ 0 & -M_i & M_j \end{bmatrix} B_{ij,k} = \begin{bmatrix} M_i & -M_i & 0 \\ M_j & 0 & M_j \\ 0 & -M_k & M_k \end{bmatrix} A_{ij,k} + Q.$$

A simple calculation shows that $B_{ij,k} = WA_{ij,k} + Q$, where

$$W = (1/2\mu) \begin{bmatrix} M_k(M_i^2 + M_j^2) & M_k(M_k^2 - M_i^2) & M_k(M_j^2 - M_k^2) \\ M_j(M_j^2 - M_i^2) & M_j(M_i^2 + M_k^2) & M_j(M_j^2 - M_k^2) \\ M_i(M_j^2 - M_i^2) & M_i(M_i^2 - M_k^2) & M_i(M_j^2 + M_k^2) \end{bmatrix},$$

so that from the definition of the $c_{ij,k}$ it follows that

$$C_{ij,k} = UA_{ij,k} + Q, \tag{2.10}$$

where

$$U = (1/2\mu) \begin{bmatrix} M_k(M_i^2 - M_j^2) & M_k(M_k^2 - M_i^2) & M_k(M_j^2 - M_k^2) \\ M_j(M_j^2 - M_i^2) & M_j(M_i^2 - M_k^2) & M_j(M_j^2 - M_k^2) \\ M_i(M_j^2 - M_i^2) & M_i(M_i^2 - M_k^2) & M_i(M_j^2 - M_k^2) \end{bmatrix}.$$

Since

$$\det U = (M_i^2 - M_j^2)(M_k^2 - M_i^2)(M_j^2 - M_k^2)/2,$$

we see that $A_{ij,k}$ and $B_{ij,k}$ are both of the form $QC_{ij,k} + Q$. It therefore follows from the definition of the $c_{ij,k}$ together with (2.2) and (2.3) that

$$a_{ij,k} = Q_c, \qquad b_{ij,k} = Q_c, \qquad 1 \leqslant i, j, k \leqslant n. \tag{2.11}$$

It is well known, and indeed easily verifiable, that

$$[D_l, D_k]u = \sum_m (a_{mk,l} - a_{ml,k}) D_m u, \tag{2.12}$$

where $[D_l, D_k]u = D_l(D_k u) - D_k(D_l u)$. Now,

$$D_l\langle D_k X_j, X_i\rangle = \langle D_l D_k X_j, X_i\rangle + \langle D_k X_j, D_l X_i\rangle + \sum_{p,q} (D_k X_j)^p D_l G_{pq} X_i^q,$$

so that by (2.1) and (2.11) it follows that $D_l\langle D_k X_j, X_i\rangle = \langle D_l D_k X_j, X_i\rangle + Q_c$. Thus we have

$$D_l\langle D_k X_j, X_i\rangle - D_k\langle D_l X_j, X_i\rangle = \langle [D_l, D_k] X_j, X_i\rangle + Q_c.$$

Therefore, upon taking into account (2.12) we conclude that

$$D_l a_{ij,k} - D_k a_{ij,l} = Q_c. \tag{2.13}$$

In precisely the same manner we obtain

$$D_l b_{ij,k} - D_k b_{ij,l} = Q_c. \tag{2.14}$$

We can now easily derive equations of the desired form for the $c_{ij,k}$. Here we use the symbol Q'_c to denote an expression of the same form as Q_c but in which in the polynomial forming the numerator we also allow second derivatives of the M_i, G_{ij}, H_{ij}. One easily sees that because of (2.11) $D_k Q = Q'_c$. We have by (2.5), (2.6), (2.13), and (2.14) that

$$\begin{aligned} D_j a_{ij,i} &= D_j((M_j/M_i) b_{ij,i} + Q) = (M_j/M_i) D_j b_{ij,i} + Q'_c \\ &= (M_j/M_i) D_i b_{ij,j} + Q'_c = (M_j/M_i) D_i((M_j/M_i) a_{ij,j} + Q) + Q'_c \\ &= (M_j/M_i)^2 D_i a_{ij,j} + Q'_c = (M_j/M_i)^2 D_j a_{ij,i} + Q'_c, \end{aligned}$$

so that $D_j c_{ij,i} = Q'_c$. Analogously, one sees that $D_i c_{ij,j} = Q'_c$ also. Now let $i > j > k$. Then

$$\begin{aligned} D_j b_{ij,k} &= D_k b_{ij,j} + Q_c = D_k((M_j/M_i) a_{ij,j} + Q) + Q_c \\ &= (M_j/M_i) D_k a_{ij,j} + Q'_c = (M_j/M_i) D_j a_{ij,k} + Q'_c, \end{aligned}$$

so that $D_j c_{ij,k} = Q'_c$. In an analogous way one sees that $D_i c_{ik,j} = Q'_c$ and $D_k c_{jk,i} = Q'_c$. In summary, we have that there are indices $\alpha = \alpha(i, j, k)$ for $i > j$, k arbitrary such that

$$D_\alpha c_{ij,k} = Q'_c. \tag{2.15}$$

To obtain equations for the variables X_i^p, Y_i^p we use the fact that $\langle D_k X_i, X_j\rangle = a_{ji,k} = Q_c$ and $\langle D_k Y_i, Y_j\rangle = b_{ji,k} = Q_c$, by (2.11). If we denote by X and Y the matrices whose columns are X_i and Y_i, respectively, then these equations say that $X^T G(D_k X) = Q_c$ and $Y^T H(D_k Y) = Q_c$, so that we have

$$D_k X = G^{-1}(X^T)^{-1} Q_c \tag{2.16}$$

$$D_k Y = H^{-1}(Y^T)^{-1} Q_c. \tag{2.17}$$

Finally, we note that since $JX_k = M_k Y_k$, we have

$$D_k f_i = M_k Y_k^i, \qquad 1 \leqslant i \leqslant n. \tag{2.18}$$

The system of first order partial differential equations for the $n(n+1)(n+2)/2$ unknown functions f_i, X_i^p, Y_i^p, $c_{ij,k}$, henceforth to be referred to as DIAG, consists of Eq. (2.15) together with the case $k=1$ of Eqs. (2.16), (2.17), and (2.18). Here the choice of $k=1$ is completely arbitrary.

3. Diagonal Hyperbolic Systems

A diagonal hyperbolic system is a set of N partial differential equations for an N-tuple $u = (u_1, ..., u_N)$ of unknown functions of $x = (x_1, ..., x_n)$ of the form

$$Z_j(u, x) \cdot \nabla u_j = F_j(u, x), \qquad 1 \leqslant j \leqslant N. \tag{3.1}$$

The system DIAG derived in the preceding section is clearly of this type, the form of the Z_j being particularly simple. Of interest to us is the initial value problem in which one is given an N-tuple $g = (g_1, ..., g_N)$ of functions on a domain D_0 of $\mathbf{R}^{n-1}$ such that

$$Z_j^{(n)}(g(x)) \neq 0, \qquad x \in D_0,\ 1 \leqslant j \leqslant N \tag{3.2}$$

and $\{(g(x), x) : x \in D_0\}$ is contained in the domain of all of the functions Z_j, F_j, and one is to find in a neighborhood of D_0 a solution u of (3.1) satisfying

$$u(x) = g(x), \qquad x \in D_0. \tag{3.3}$$

The conditions (3.2) stipulate that the initial data are non-characteristic, and allow one to rewrite the system in the form

$$\frac{\partial u_j}{\partial t} + \sum_{i=1}^{n-1} \lambda_{ij}(u, \bar{x}, t) \frac{\partial u_j}{\partial x_i} = C_j(u, \bar{x}, t), \tag{3.4}$$

where $\bar{x} = (x_1, ..., x_{n-1})$ and $t = x_n$. The initial value problem (3.3), (3.4) may be treated by the well known method of Courant and Lax [CL] which is based on the observation that for C^1 functions it is equivalent to a system of integral equations in which the set of unknown functions consists of the u_j together with a set of N $(n-1)$-dimensional vector functions $X_j(t; \bar{\xi}, \tau)$ used to give the characteristic curves through $(\bar{\xi}, \tau)$ corresponding to a solution u of (3.4). The book of Garabedian [Gar, Sect. 4.3] gives careful exposition of the ideas involved in the case $n=2$; the presence of a greater number of independent variables adds no complication. We briefly indicate the essence of the argument. The integral equations are

$$\begin{aligned} u_j(\bar{\xi}, \tau) &= g_j(X_j(0; \bar{\xi}, \tau)) + \int_0^\tau C_j(u(X_j(\sigma; \bar{\xi}, \tau), \sigma), X_j(\sigma; \bar{\xi}, \tau), \sigma)\, d\sigma \\ &= I_u(u, X) \end{aligned} \tag{3.5}$$

$$\begin{aligned} X_j(t; \bar{\xi}, \tau) &= \bar{\xi} + \int_\tau^t \lambda_j(u(X_j(\sigma; \bar{\xi}, \tau), \sigma), X_j(\sigma; \bar{\xi}, \tau), \sigma)\, d\sigma \\ &= I_X(u, X), \end{aligned} \tag{3.6}$$

where $\lambda_j = (\lambda_{ij}, ..., \lambda_{(n-1)j})$. Let E denote the (common) domain of the functions λ, C, let L_e be a Lipschitz constant for these functions on E, let L_g be a Lipschitz constant for g on D_0, and let $d(a, B)$ denote the distance from point a to set B. For $\bar{x} \in \mathbf{R}^{n-1}$, let $P(\bar{x}, h, K)$ denote the double pyramid

$$P(\bar{x}, h, K) = \{(\bar{\xi}, t) : |\xi_i - x_i| < (h - |t|)K,\ 1 \leqslant i \leqslant n-1\}.$$

By a straightforward iteration of the integral operators I_u and I_X with initial iterates $u^0(\bar{\xi}, \tau) = g(\bar{\xi})$ and $X_j^0(t; \bar{\xi}, \tau) = \bar{\xi}$, one can show that there are numbers h, K, and L which depend only on n, N, L_c, L_g, $d(\bar{x}, \partial D_0)$, and $d((g(\bar{x}), \bar{x}, 0), \partial E)$ such that the system (3.5), (3.6) possesses a unique solution in $P(x, h, K)$ which is Lipschitz continuous with Lipschitz constant L there. In particular, this system has a unique solution in the union U of these pyramids. In an obvious manner one can extend the solution bit by bit to all points in $\mathbf{R}^n$ whose domain of dependence is contained in D_0 and at which the solution u is locally Lipschitz continuous. If one assumes that the initial data as well as λ and C are C^2, then one can easily show that this solution u of (3.5), (3.6) is actually a continuously differentiable solution of (3.3), (3.4).

For the purposes at hand we need one additional property of diagonal hyperbolic systems, namely, that if λ and C are analytic in E and the initial data g are analytic in D_0, then the solution u of (3.5), (3.6) obtained by the method just sketched is actually analytic in U. The simplest way to convince oneself of the validity of this conservation of analyticity principle is to replace $\bar{x}$ in the foregoing by an $(n-1)$-tuple of complex numbers $\bar{z}=(z_1, ..., z_{n-1})$ and to observe that the operators I_u, I_X preserve analyticity with respect to these $n-1$ variables. One now shows that the iterates are well defined and converge uniformly to a function w in a pyramid in $\mathbf{C}^n \times \mathbf{R}$ of the form

$$\{(\bar{\zeta}, t) : |\mathrm{Re}\, \zeta_i - x_i| \leqslant (h-|t|)K,\ |\mathrm{Im}\, \zeta_i| \leqslant (h-|t|)\, \delta K,\ 1 \leqslant i \leqslant n-1\}.$$

Here h, K are as before, but $\delta > 0$ will depend on g. The resulting function w will be complex analytic in $z_1, ..., z_{n-1}$ in this pyramid, and therefore $u(\bar{x}, t) = w(\bar{x}, t)$, in addition to being a solution of (3.5), (3.6), will be real analytic in $P(x, h, K)$. A simple argument based on uniqueness of the solution of (3.5), (3.6) together with the Cauchy–Kowalewski theorem will then show that $u(\bar{x}, t)$ is real analytic in all n variables in U. It is worth noting that this analyticity principle can also be established by considering an appropriate real diagonal hyperbolic system in which the $2N$ unknown functions of $2n-1$ variables are the real and imaginary parts of the w_j, the point being that the Cauchy Riemann equations propagate along characteristics. The ideas underlying such an approach are implicit in the discussion to be found in [Gar, Sect. 16.1].

The theory we have just outlined can be applied to the DeTurck–Yang initial value problem of Sect. 1: Assume that G, H, M, and the initial data ϕ are all C^4 and let $\{G^s\}$, $\{H^s\}$, $\{M^s\}$, and $\{\phi^s\}$ be sequences whose elements have polynomial components and which converge uniformly, together with their derivatives of order less than or equal to four to G, H, M, and ϕ and their corresponding derivatives. For any fixed σ and each s, we calculate the values of X, Y, and c on D_0 using the system (1.7) with G^s, H^s, M^s, and ϕ^s and denote the resulting functions by ξ^s, η^s, and γ^s. Since as a consequence of the comment at the end of the next to last paragraph of Section 1 the x_n-components of the vectors ξ_i^s are bounded away from 0, it follows from the foregoing discussion that there is a neighborhood R of D_0 in $\mathbf{R}^n$ which is independent of s and in which there exists an analytic solution f^s, X^s, Y^s, c^s of DIAG with initial values given on D_0 by ϕ^s, ξ^s, η^s, γ^s, and furthermore that these solutions are uniformly Lipschitz continuous on R. As pointed out at the end of Section 1, for each s there is an analytic solution $\tilde{f}^s$ of (1.7) in some neighborhood R^s of D_0 which coincides with ϕ^s on D_0. The calculations of Section 2, however, show that $\tilde{f}^s$ and its accompanying functions $\tilde{X}^s$, $\tilde{Y}^s$, $\tilde{c}^s$ also satisfy (2.15) through (2.18), and in particular are a solution in R^s of DIAG with initial data ϕ^s, ξ^s, η^s, γ^s. From the uniqueness of the solution of the initial value problem for diagonal hyperbolic systems, it follows that $\tilde{f}^s = f^s$ in $R \cap R^s$. It then follows from the principle of permanence of functional equations for analytic functions that f^s must satisfy (1.7) in all of R. Since f^s must converge uniformly on R (to a solution of (1.7) corresponding to ϕ, G, H, M), we conclude that if the functions G, H, M as well as the initial data are of class C^4, then the DeTurck–Yang initial value problem is locally solvable. What is more, it is clear from the above discussion that one can obtain a "lower bound" for the neighborhood of D_0 in which the solution exists in terms of bounds on appropriate derivatives of G, H, M, ϕ and lower bounds on the differences $K_i - M_i$, $M_{i+1} - K_i$, $1 \leqslant i \leqslant n-1$, the K_i being the principal strains of the initial mapping ϕ as described in Section 1.

By taking into account the special form of the system DIAG we can show that there exists a local solution to the DeTurck–Yang initial value problem under the weaker hypothesis that the initial data have locally Lipschitz first derivatives. (Although the regularity requirements on G, H, and M can also be weakened somewhat, we do not pursue this point and continue to assume that they are of class C^4.) To see this, consider a sequence $\{\phi^s\}$ of C^4 mappings which converge uniformly to ϕ and whose first partial derivatives are uniformly Lipschitz continuous, that is, whose second partial derivatives are uniformly bounded. For fixed σ and each s we calculate ξ^s, η^s, and γ^s as before. Clearly, the ξ^s and η^s are uniformly Lipschitz continuous and the γ^s are uniformly bounded. For each s, wherever the corresponding solution f^s, X^s, Y^s, c^s of DIAG exists, the identities (1.2), (2.16), (2.17), and (2.18) will hold, as follows from the considerations of the preceding paragraph. Now, a bound on γ^s clearly gives a bound for c^s and consequently for the first derivatives of X^s, Y^s via (1.2), (2.16), and (2.17). Because of the form of DIAG and the special nature of the initial conditions we can show that no matter how large the initial values of the first derivatives of c^s may be, they will not blow up before some component of c^s does. To understand why this is so, one applies (2.12) with $u = c_{ij,k}$ to (2.15) to see that

$$D_\alpha D_\beta c_{ij,k} = D_\beta Q'_c + \sum_m (a_{m\beta,\alpha} - a_{m\alpha,\beta}) D_m c_{ij,k},$$

and therefore because of (2.16), (2.17), and (2.18)

$$D_\alpha(D_\beta c_{ij,k}) = \sum_{mijk} S_{mijk} D_m c_{ij,k} + T, \tag{3.7}$$

where β is arbitrary, α is as in (2.15), and the coefficients S_{mijk}, T are expressions involving f, X, Y, c together with G, H, M, and their derivatives of order less than or equal to 3. The linear nature of (3.7) permits one to obtain the desired a priori bound for all first derivatives of c. In this manner one can see that there is a domain R, independent of S on which all of the solutions f^s, X^s, Y^s, c^s exist, its size depending on an upper bound for γ^s (as well as, of course, on bounds on appropriate derivatives of M, G, H, and on the differences $K_i - M_i$, $M_{i+1} - K_i$). Since the sequences $\{X^s\}$, $\{Y^s\}$ are uniformly Lipschitz continuous, $\{f^s\}$ has a subsequence which converges uniformly to the desired mapping in R.

It is in addition clear from the argument we have just outlined that because of the relationship (2.10) between the $c_{ij,k}$ and the $a_{ij,k}$, a solution of the DeTurck–Yang initial value problem will break down only when one of the $a_{ij,k}$'s blows up, that is, when a singularity arises in the curvature of one of the characteristic curves or in the rate at which the frame determined by $n-1$ of these curves varies as one moves along the remaining curve. This is quite analogous to the way in which singularities appear in one dimensional wave phenomena (where, loosely speaking, the only possibility is for blow-up of a curvature since the underlying space is $\mathbf{R}^2$). It would be of interest to derive non-trivial sufficient conditions on an initial mapping ϕ of $\mathbf{R}^{n-1}$ which would permit one to establish global existence by precluding such blow-up, at least in the simplest context in which G and H are the identity matrix and the M_i are constant. In this case it is not hard to show that when $n=2$ there exist no C^2 global solutions other than affine mappings (no matter what ϕ is); in three or more dimensions, however, there are non-trivial solutions as shown by Yin [Y], so that the pursuit of such criteria might provide a fruitful line of investigation.

We close by mentioning that DeTurck and Kamberov [DK] and Kamberov [K] have studied a "singular" version of the DeTurck–Yang initial value problem in which two of the M_i coincide on the initial manifold and by suggesting the possibility that this problem might also be treated by means of the system DIAG.

References

[CL] R. Courant and P. D. Lax, On nonlinear partial differential equations with two independent variables, *Comm. Pure Appl. Math.* **2** (1949), 255–273.

[DK] D. DeTurck and G. Kamberov, A singular problem in elasticity theory, *Contemp. Math.* **68** (1987), 77–84.

[DY] D. DeTurck and D. Yang, Existence of elastic deformations with prescribed principal strains and triply orthogonal systems, *Duke Math. J.* **51** (1984), 243–260.

[Gar] P. Garabedian, "Partial Differential Equations," Wiley, New York, 1964.

[Gas] J. Gasqui, Sur l'existence locale de certaines métriques riemanniennes plates, *Duke Math. J.* **45** (1979), 109–118.

[K] G. Kamberov, Prescribing the strain of a diffeomorphism and solvability of the singular Cauchy problem, preprint, 1990.

[Y] W.-L. Yin, Two families of finite deformations with constant strain invariants, *Mech. Res. Comm.* **10** (1983), 127–132.